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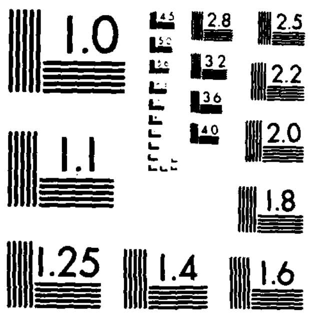

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NRL Memorandum Report 4918

Efficient Elastic-Plastic Design of Small Foundations

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An elastic-plastic design analysis method is presented for small foundations which utilizes an energy criterion to insure against large deflections or unwanted collapse. The interaction effects from direct loads and shears are included, which has the effect of predicting a carrying capacity somewhat less than that of ordinary limit analysis. The energy storage capacity is calculated in a fashion which is conservative since the reduction of the planar moments of inertia is ignored. Several examples are worked out which use various beam like structures to illustrate the

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20 ABSTRACT (Continued)

enhancement of carrying capacity from the sections contribution and of the indeterminate structural reactions. The results are worked up in detail, both theoretically and by numerical examples to show the ease of application and efficiency of this method.

CONTENTS

EFFICIENT ELASTIC-PLASTIC DESIGN OF SMALL FOUNDATIONS

INTRODUCTION

An energy method has been reported earlier [1] that describes an efficient elastic design method for small foundations on shipboard structures. This method may be used to circumvent the Navy's Dynamic Design Analysis Method (DDAM) in those cases where a single degree of freedom (SDOF) system represents the equipment foundation structure. The assumption of a SDOF system is equivalent to saying that the static and dynamic stress and deflection patterns are the same so that a direct approach taking advantage of this can lead to an efficient design method that utilizes energy criteria.

The purpose of this report is to extend the energy method so that the foundation material undergoes elastic-plastic behavior. This new elastic-plastic energy method utilizes both limit analysis and the interaction effects between bending and shear as developed in plasticity theory. It is shown through an example problem that the method provides foundation designs that can lead to substantial weight and cost reductions.

For purposes of this report and its sample calculations the design is based upon the following hypothetical description of the shock environment: "The equipment foundation system shall be designed to withstand elastic-plastically the lesser of a 250-g equivalent static acceleration or a shock spectrum pseudo-velocity, V_A, of 8 ft/sec. The structure shall not experience a collapse mechanism so that all deflections remain small."

ENVIRONMENTAL DESCRIPTION

One of the vexing problems with DDAM has been the need to interpret measured shock response for purposes of developing design shock spectra. A significant contribution to this problem has been reported earlier [2]. It was shown that when equipment-foundation combinations yield under a shock input, the design shock spectra obtained is greater than the elastic design shock spectra.

To demonstrate this concept, consider Fig. 1 which shows an equipment that is attached to a vehicle by a foundation. This equipment-foundation structure is subjected to some shock input. Imagine that five foundations, identical except for their yield points, are to be each inserted and the combination is subjected to the same shock source. Since each has the same weight and same fixed base frequency, the elastic shock design value is the same in each case [3]. Also, assume that if the structure remained elastic, a 90 ksi stress would occur in the foundation.

The results might appear as in Fig. 2. This shows the shock design value versus fixed base frequency. Note that when the foundation was made of 30, 50, or 80 ksi material, the apparent shock design values are greater than when it is either 100 or 150 ksi. Therefore, the elastic design value is a minimum so that the two high strength foundations remained elastic (maximum stress of 90 ksi), and the other three foundations went plastic. Thus, the weaker foundations give higher values for an elastic design. In other words, if a system were built of 100 ksi steel on the basis of the 50 ksi steel test, the foundation would be overdesigned.

Manuscript submitted August 12, 1982

LIMIT ANALYSIS

A. Bending Effects

By way of introduction to limit analysis, consider the case of perfectly plastic behavior of a material by examining the stress distribution across a beam cross-section due to bending only as shown in Fig. 3. Figure 3(a) shows the elastic stress distribution where Hooke's law holds and the tensile and compressive yield stresses are equal to each other. Figure 3(b) shows the elastic-plastic stress distribution as the applied load to the beam is increased. Here, the outer portions of the beam fibers have reached the elastic limit while the inner core of the beam remains elastic. Finally, as the load is increased further, the stress distribution reaches the fully plastic condition shown in Fig. 3(c). In this condition a plastic hinge is assumed to exist at the cross-section and the loads that just produce this condition are called the collapse loads.

Consider the cantilever beam with a rectangular cross-section subjected to the concentrated load P as shown in Fig. 4(a). The plastic hinge will form at the root of the beam where the maximum bending stresses occur. The load deflection relationship for this condition has been developed for the case of perfectly plastic behavior [4] and is sketched in Fig. 4(b). We observe that the collapse load P_c is 1.5 times greater than the elastic limit load P_e. The corresponding deflection when the collapse mechanism is just reached is 2.22 times the elastic limit deflection δ_e. The increase in the load P in going from the elastic limit to fully plastic behavior is identified as the load enhancement, while the area under the load-deflection curve AB represents the energy enhancement as shown in Fig. 4(b). Of course, in an actual design we wish to stop short of reaching the collapse load. For a design method that utilizes energy, one approach for a conservative estimate of the energy enhancement would be to calculate the area under the curve AC. In effect we would be assuming that the beam remains elastic out to the collapse load. In reality, the beam behavior has been extended into the elastic-plastic region but short of reaching the collapse mechanism.

As a final comment on Fig. 4, it is noted that if the energy absorbed by the cantilever beam exceeds the area under the load-deflection curve OAB, the danger of an instability leading to the beam collapse exists.

In addition to the cantilever beam in Fig. 4, other examples of beams subjected to a variety of loading conditions indicate that deflections due to bending remain relatively small (on the order of four to five times the elastic limit deflections) until near the collapse load. Consequently, if a procedure can guarantee that the collapse mechanism is not fully developed, then a design would be valid and weight and cost reductions would follow.

B. Propped Cantilever Beam

A second factor that contributes to load enhancement is to design the beam as a statically indeterminant structure. This leads us to consider the propped cantilever beam in Fig. 5 that carries two concentrated forces both of magnitude P. It is not readily apparent where the plastic hinges occur in order to carry out a limit analysis of the beam. A complete analysis is found in Appendix A which provides the correct plastic hinges that are shown in Fig. 6. The corresponding loading diagram, shear diagram, and bending-moment diagram for this condition are shown in Figs. 7(a), (b), and (c), respectively, which we shall examine in greater detail in the example problem.

STRESS RESULTANT THEORY

The formation of the plastic hinges for the propped cantilever beam in Fig. 6 was based upon the effect of bending only. In reality, there is both shear and bending present throughout the beam as shown in Fig. 7. Shear effects become especially important for short deep beams and should not be

overlooked in the analysis. To account for the presence of both the shear force and bending moment at the highly stressed sections of the beam, the following interaction expression, that is developed in Appendix B, is applicable:

$$\left|\frac{M}{M_o}\right|^2 + \left|\frac{S}{S_o}\right|^2 = 1 \qquad (1)$$

where M = bending moment present

S = shear force present

M_o = limiting value of the bending moment in the absence of shear

S_o = limiting value of the shear in the absence of bending

This equation, therefore, will determine the carrying capacity of the beam as demonstrated in the example problem.

DESIGN CRITERIA

Consider the [illegible] system attached to a [illegible] as shown in Fig. 6. Let W be the weight of the equipment and an assigned portion of the foundation weight. The foundation needs to withstand the [illegible] of 250 g's or a pseudo velocity $V = 2\pi f \delta = 6$ ft/sec, where δ = relative displacement between the weight and the base.

The 250 g design check is relatively straight forward as shown in the example problem.

In the case of the velocity input, the kinetic energy of the [illegible] system represents the energy absorbed by the foundation which is labelled E_a. If E_s is the energy storage capacity of the foundation, then the survival criteria for the equipment-foundation combination is as follows:

$$E_a \le E_s \qquad (2)$$

That is, if the actual energy stored in the foundation is less than or equal to the energy storage capacity of the foundation, the structure will survive the mechanical shock. It is assumed that no energy is stored in the equipment.

Suppose the foundation consists of a beam that is subjected to a combination of bending, shear and direct (axial) loads. The energy absorbed by the beam for each type of load is represented as follows:

$$\text{Bending Energy} = U_b = \int_0^l \frac{M^2\,dx}{2EI} \qquad (3)$$

$$\text{Shear Energy} = U_s = \int_0^l \frac{\alpha S^2\,dx}{2AG} \qquad (4)$$

$$\text{Direct Energy} = U_d = \int_0^l \frac{P^2\,dx}{2AE} \qquad (5)$$

where M = bending moment, S = shear, P = axial force, E = Young's modulus, I = moment of inertia, A = cross-sectional area, G = shear modulus, α = ratio of A to the web area, and l = length of the beam. The total available storage energy is

$$E_s = U_b + U_s + U_d \qquad (6)$$

In the example problem developed in this report, only bending and shear will be present, so that

$$E_s = U_b + U_s \qquad (7)$$

It is also noted that Eq. (4) provides shear energy that closely follows a theory illustrated by Timoshenko [5].

The calculation for the bending energy represented by Eq. (5) does not take into account the reduced moment of inertia, I_r, that exists in the beam as discussed in Appendix C, i.e.

$$\int_0^L \frac{M^2 dx}{2EI_r} < \int_0^L \frac{M^2 dx}{2EI} \qquad (8)$$

where $I_r < I$. Furthermore, by setting $\alpha = 1$ in Eq. (4), we shall obtain a smaller shear energy, i.e.

$$\int \frac{V^2 dx}{2\,GA_s} < \int \frac{\alpha\,V^2 dx}{2\,GA_s} \qquad (9)$$

Consequently, the proposed analysis provides an energy storage capacity that is less than the amount that is available, and hence the design approach is on the conservative side.

EXAMPLE PROBLEM

1. Foundation Description

Consider the two propped cantilever beams in Fig. 9 that form the foundation for a rigidly attached piece of equipment. Each beam is a [illegible] WF [illegible] structural member and it is assumed that they equally share the load in vertical shock. The material is steel with a tensile and compressive yield point of 30 ksi and a shear yield point of [illegible] ksi. The problem is to find the carrying capacity of these beams for the shock environment described in the Introduction.

Figure 10 shows the cross-section of a typical [illegible] WF [illegible] member. Figure 11 is a model of one of the propped cantilever beams in which loads are shown acting at the points of attachment between the equipment, whose total weight equals W, and the foundation. Replace each equipment load with P as shown in the loading diagram in Fig. 7, where P now represents the maximum allowable force that the foundation can withstand for elastic-plastic response to severe under-water shock loading.

Recall that Fig. 8 shows the location of the plastic hinges due to bending effects only for the propped cantilever beam. It is interesting to calculate the collapse loads using the data from the bending and shear diagrams of Figs. 7(b) and 7(c) in Eq. (11). For example, at station 1, $V = 1.[illegible]\,P$ and $M = [illegible]\,P$. Now $V_y = [illegible]$ lb and $M_y = [illegible]$ in-lb. Substituting into Eq. (1) gives $P = [illegible]$ lb. Just to the left of station 2 where the other plastic hinge is located, $V = [illegible]\,P$ and $M = [illegible]\,P$. Now Eq. (1) leads to a collapse load $P = [illegible]$ lb. It is not surprising that two different collapse loads were found since the shear and bending moment diagrams were based upon the application of limit analysis that addressed bending effects only.

The next step is to consider Fig. 12 that shows a cut section just to the left of station 2. Now at the cut section the shear $V = Q$ and the bending moment $M = 30\,Q$. Substituting into Eq. (1)

$$\left[\frac{30\,Q}{M_y}\right]^2 + \left[\frac{Q}{V_y}\right]^2 = 1$$

Solving for Q yields

$$Q = \frac{M_y\,V_y}{\left[900\,V_y^2 + M_y^2\right]^{1/2}} = [illegible]\ \text{lb}$$

as the maximum possible reaction.

At the built-in end the moment $M = [illegible]\,P - [illegible]\,Q$ and the shear is $V = [illegible]\,P - Q$. Substituting into Eq. (1)

$$\left[\frac{[illegible]\,P - [illegible]\,Q}{M_y}\right]^2 + \left[\frac{[illegible]\,P - Q}{V_y}\right]^2 = 1$$

Solving for P yields

$$P = 62{,}[illegible] \text{ lb}$$

[illegible] $P = 62{,}[illegible]$ lb.

B. Energy Storage Capacity

[illegible]

$$E_{[illegible]} = [illegible] \text{ in-lb}$$

$$E_{[illegible]} = [illegible] \text{ in-lb}$$

[illegible]

$$E = [illegible] \text{ in-lb}$$

[illegible]

[illegible]

$$[illegible]$$

Recall that [illegible]. Thus,

$$E = \frac{1}{2} [illegible] \quad (10)$$

[illegible]

$$[illegible] \quad (11)$$

DESIGN REQUIREMENTS

A. Equivalent Static Design

Recall that [illegible] P. [illegible]

$$[illegible] \quad (12)$$

where W = design load [illegible]

$$[illegible]$$

or

$$[illegible] \quad (13)$$

For the case of N = 250 g's and a collapse load P = 62,600 lb,

$$w = \frac{62{,}600}{250} - 62.5 = [illegible]$$

The total weight that must be carried [illegible] [illegible] the foundation weighs [illegible] lb, then the weight of the foundation is [illegible]% of the weight it carries. Note that this design allows the foundation to be fully plastic [illegible]

B. Energy Design

[illegible]

Bending Energy Only

[illegible]

[illegible]

Therefore, [illegible]

It is interesting to compute the number of g's required to [illegible] this weight. Thus,

[illegible]

Bending and Shear Energy

Now let [illegible]

[illegible]

Since the total weight [illegible] the weight of the foundation is [illegible]% of the weight it carries. The equivalent "g" number is

[illegible]

C. Effective Design Velocities, V_e

Equation [illegible] is now rearranged to find the effective design velocities as follows.

[illegible] ([illegible])

where [illegible] = total energy storage capacity of the design

[illegible]

[illegible] ([illegible])

Since w = 333.7 lb was calculated for the design based on bending energy only,

$$V_e = 6.30 \text{ ft/sec}$$

Thus, if the beam were designed to withstand 6 ft/sec and bending energy alone were used to establish the critical weight, the beam would not develop a collapse mechanism. Consequently, the beam deflections would be small.

Likewise, it is interesting to apply Eq. [illegible] for the 250-g design where w = 100.2 lb. Now V_e = 11.03 ft/sec.

D. Efficiency Factor

The following relationship is a measure of the design efficiency in terms of energy:

$$\text{Efficiency Factor} = \sqrt{\frac{\text{actual energy absorbed}}{\text{total energy capacity}}} \qquad (16)$$

For the example problem under consideration, the total energy capacity = E = [illegible] in-lb. For the design that utilized the bending energy only, E_b = [illegible] in-lb so that

$$\text{Efficiency Factor} = \sqrt{\frac{[illegible]}{[illegible]}} \times 100 = [illegible]$$

In the case of the 250-g design, w = 100.2 lb. The corresponding energy absorbed is

$$E = [illegible]$$

Therefore

$$\text{Efficiency Factor} = \sqrt{\frac{[illegible]}{[illegible]}} \times 100 = [illegible]$$

It is interesting to observe that these values are related to the ratios of the velocities V_e's.

RESULTS

The results of the elastic-plastic design method are summarized in Table I using the following notation:

- WCT — the total carried weight (lb) by one grouped beam
- % — the percent of the foundation weight to the carried weight
- G — the g design value
- V_e — the effective velocity design value (ft/sec) for the configuration based upon the structure's ability to absorb energy

The expressions in the first column indicate the method by which WCT was computed.

An elastic design method was reported earlier [12] in which the same example problem of the grouped cantilever beams was examined. The load P = 27,200 lb was the maximum design load that assured that the grouped beams would remain elastic. This value compares with the load P = 67,600 lb reported herein for the plastic behavior of the beams. Table II summarizes the results for these two design methods as a convenient basis for comparing the results in terms of WCT and the efficiency factor. These results show the dramatic improvement in load carrying capacity of the foundation when elastic-plastic behavior occurs. It is especially interesting to observe that the maximum carried weight WCT calculated by the elastic-plastic design method is 7.3 times larger than the largest WCT calculated by the elastic design method.

Table I — Summary of the Elastic-Plastic Design Method
P = 62,680 lb, $\mathcal{E}_1$ = 9,926 in.-lb, $\mathcal{E}$ = 11,374 in.-lb

Failure Criteria	WGT (lb)	[illegible]	S (g)	V_e (ft/sec)	Efficiency Factor
250 g Design	376.4	49.8	250	11.03	72.5
Bending Energy	707.4	26.5	150.6	8.56	93.4
Total Energy	828.8	22.6	131.4	8.00	100

Table II — Summary of the Elastic Design Method
and the Elastic-Plastic Design Method

Design Method	Design Load P (lb)	Failure Criteria	WGT (lb)	Efficiency Factor
Absolute Elastic	35,310	250-g Design	157.4	54.4
		Bending Energy	110.4	49.7
		Total Energy	242.0	62.0
Elastic Plastic	62,680	250-g Design	376.4	72.5
		Bending Energy	707.4	93.4
		Total Energy	828.8	100

A final comparison between the elastic design method and the elastic-plastic design method is shown in Table III. This table lists all of the remaining pertinent data obtained by the two methods for completeness. The additional notation includes the following:

FREQ — the Rayleigh estimate for the fixed base frequency (Hz)

S ELASTIC — the g-design value for the beam to remain elastic under the load WGT

S FAILURE — the g-design value for the beam to experience some plastic behavior under the load WGT

V ELASTIC — the effective velocity design value (ft/sec) for the configuration to remain elastic based upon the structure's ability to absorb energy

V FAILURE — the effective velocity design value (ft/sec) for the configuration to experience some plastic behavior under the load WGT

WGT/WMAX — the ratio of the total carried weight by one propped beam to the maximum carried weight using the total energy, elastic-plastic design method

Table III — Summary

Failure Criteria	Design Approach	WGT	Freq	[illegible]	N Elastic	N Failure	V Elastic	V Failure	WGT WMAX	Comment
Allowable Elastic	30 g	157.4	140.5	119.1	250.0	443.8	9.12	14.70	0.190	Overdesign
	Bending Energy	110.4	153.9	169.8	300.0	532.5	9.99	16.10	0.133	Overdesign
	Total Energy	242.0	123.3	77.5	192.4	341.5	8.00	12.90	0.292	Efficient Elastic
Shear at C.L. Web	30 g	215.0	128.1	87.2	207.7	368.7	8.31	13.40	0.259	Elastic
	Bending Energy	216.2	127.8	86.7	207.0	367.5	8.30	13.38	0.261	Elastic
	Total Energy	406.6	102.4	46.1	132.8	235.7	6.65	10.72	0.491	Some Plasticity
Nominal Bending Elastic	30 g	227.4	125.6	82.5	200.4	355.7	8.16	13.16	0.274	Elastic
	Bending Energy	241.4	123.3	77.7	192.7	342.1	8.00	12.91	0.291	Elastic
	Total Energy	446.2	98.8	42.0	123.6	219.4	6.41	10.34	0.538	More Plasticity
Combined Stress Resultant	30 g	376.4	105.5*	49.8	140.8*	250.0	6.84*	11.03	0.454	
	Bending Energy	707.4	81.9*	26.5	84.8*	150.6	5.31*	8.56	0.854	
	Total Energy	828.8	76.5*	22.6	74.0*	131.4	4.96*	8.00	1.000	Large AMT Plasticity

*These can be found in this case because the elastic solution is known.

SUMMARY

The following summarizes the major assumptions associated with the proposed elastic-plastic design method:

1. The equipment-foundation combination is modeled as a SDOF system.

2. The foundation behaves as an elastic perfectly plastic material.

3. Limit analysis is applicable for finding the plastic hinges. The collapse load is modified by Eq. (11) to account for the interaction effects between shear forces and bending moments.

4. The calculated energy storage capacity of the foundation is underestimated as noted by Eqs. (8) and (9).

5. The equipment does not absorb energy.

6. The collapse mechanism is not formed so that deflections remain small.

The proposed method has been developed as a practical approach for designing lightweight foundations for equipment subjected to a shock environment. The method is an extension of the elastic design method reported earlier [1]. An appreciable improvement is experienced in the allowable equipment weight that may be carried under vertical shock loading when the structural foundation undergoes some plastic deformation. Another viewpoint of the results is the corresponding dramatic weight reductions of foundations that are possible by the proposed elastic-plastic design method without compromising the shock design goals.

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Appendix A
LIMIT ANALYSIS OF A PROPPED CANTILEVER BEAM

Consider the propped cantilever beam in Fig. A-1 that is 90 units long and is made of an elastic perfectly plastic material. It is loaded at its third points by equal concentrated loads P and we wish to establish the beam's load carrying capacity using limit analysis.

For this beam two plastic hinges will form when the beam experiences a collapse mechanism. It is assumed that only bending contributes to the formation of these plastic hinges.

Assume hinges develop at points 1 and 2. Using the Principle of Virtual Work and the collapse mechanism of Fig. A-2a,

$$\text{Work of Forces} = P\frac{\delta}{2} + P\delta = \frac{3P\delta}{2} = 90\ P\theta$$

and the

$$\text{Work of Moments} = M\theta + M(2\theta) + M(2\theta) = 5\ M\theta.$$

Equating these work expressions,

$$M = 18\ P.$$

The structure is now statically determinate and the reactions are calculated as shown in Fig. A-2b. The moment diagram is shown in Fig. A-2c where we observe that the moment at point 3 exceeds the calculated collapse moment of 18 P. Therefore, the true collapse mechanism was not found. However, the force $P = M/24$ must be a lower bound to the true collapse mechanism.

If all of the moment values of Fig. A-2b were reduced proportionately such that at point 3 the maximum moment is 18 P, then the statically admissible moment diagram results as shown in Fig. A-2c. However, there is only one hinge (at point 3) which is not sufficient to cause the beam to collapse. This must be, however, another bound so that

$$\frac{M_p}{24} < P_c < \frac{M_p}{18} \tag{A.1}$$

where P_c = collapse load

M_p = plastic hinge moment.

Or, in equation form,

$$P_c = \frac{(7 \pm 1)M_p}{144}. \tag{A.2}$$

Next, assume hinges develop at points 1 and 3 as shown in Fig. A-3a. Now

$$\text{Work of Forces} = P\delta + P\frac{\delta}{2} = \frac{3P\delta}{2} = 90\ P\theta$$

and the

$$\text{Work of Moments} = M(2\theta) + M(\theta) + M(\theta) = 4M\theta.$$

Equating once again,

$$M = 22.5\ P$$

The structure is now statically determinate and the reactions can be calculated with the results shown in Fig. A-3b. The shear and moment diagrams are shown in Figs. A-3c and 3d. Note that the maximum moments occur at the locations of the assumed hinges so that the true collapse load has been found. Returning to Eq. (A.1) and inserting the true collapse load,

$$\frac{M_p}{24} < \frac{M_p}{22.5} < \frac{M_p}{18}$$

which shows that the previous calculations for the assumed collapse mechanism in Fig. A-2 were indeed upper and lower bounds.

Appendix B
ELASTIC-PLASTIC INTERACTION EFFECTS

An examination of the literature dealing with the interaction effects among axial forces, shear forces, and bending moments in the plastic analysis of framed structures [4,6-9] reveals considerable activity over twenty years ago. It is apparent that no definitive result for this complex problem has evolved to date. However, there is sufficient information available from which the following conservative interaction relationship has been developed:

$$\left(\frac{M}{M_p}\right)^2 = \left[1 - \left(\frac{D}{D_p}\right)^2\right]\left[1 - \left(\frac{D}{D_p}\right)^2 - \left(\frac{S}{S_p}\right)^2\right] \tag{B.1}$$

where

M = bending moment present
M_p = limiting value of the bending moment in the absence of shear and axial forces
D = axial force present
D_p = limiting value of the axial force in the absence of bending and shear
S = shear force present
S_p = limiting value of the shear force in the absence of axial forces and bending

It is interesting to observe that if the axial force is not present, Eq. (B.1) reduces to the following expression proposed by Hodge [7] between bending and shear:

$$\left(\frac{M}{M_p}\right)^2 = 1 - \left(\frac{S}{S_p}\right)^2. \tag{B.2}$$

Also, if the shear force can be neglected, Eq. (B.1) reduces to the relationship proposed by Phillips [4] when the bending moment and axial force are present:

$$\left(\frac{M}{M_p}\right) = \left[1 - \left(\frac{D}{D_p}\right)^2\right] \tag{B.3}$$

Figure B-1 is a sketch of Eq. (B.1). Note that the type of curve on each of the three orthogonal planes are labeled, namely, a circle on the moment-shear force plane, a circle on the axial force-shear force plane, and a parabola on the moment-axial force plane. While Eq. (B.1) does not appear in the literature, each of these two-dimensional cases has been proposed by other researchers.

Appendix C
REDUCED MOMENT OF INERTIA

As the loads on a structure approach the collapse load condition, the plastic region grows along the length of the structural member where the plastic hinge(s) will ultimately form. For example, Fig. (C-1) shows a simply supported beam of rectangular cross-section loaded by a concentrated load P in case A and a uniformly distributed load q in case B. Assuming that the plastic zone covers the upper and lower quarter of the cross-section at the center of each beam, and neglecting shear effects, we observe that the elastic-plastic zones measure 0.272 L for case A and 0.522 L for case B as indicated by the shading of the beam. Consequently, for the elastic-plastic conditions shown in Fig. (C-1) we have a reduced moment of inertia I_r present in the plastic zones. It can be shown that [4]

$$I_r = 2I_e + 2\, y_e\, S_p \qquad \text{(C.1)}$$

where

- $2I_e$ = moment of inertia of the elastic region of the cross-section with respect to the neutral axis; note that this is less than the moment of inertia of the total depth of the beam when it is fully elastic
- y_e = distance of the elastic-plastic boundary from the neutral axis of the beam
- S_p = statical moment of the plastic region below the neutral axis taken with respect to the neutral axis of the beam

For example, the reduced moment of inertia at the center of the beams undergoing bending effects only in Fig. (C-1) is obtained as follows:

$$2I_e = \frac{bh^3}{96} \qquad S_p = \frac{3\,bh^2}{32} \qquad y_e = \frac{h}{4}.$$

Substituting into Eq. (C.1),

$$I_r = \frac{11}{192}\, bh^3.$$

This compares with the fully-elastic moment of inertia of

$$I = \frac{1}{12}\, bh^3$$

or a 31.25% reduction in the moment of inertia at the center of the beam.

The reduced moment of inertia phenomenon, which should include the interaction effects between shear and bending, is not treated in the calculations for the bending energy stored in the propped cantilever beams. This means that

$$\int_0^L \frac{M^2\,dx}{2\,EI} < \int_0^L \frac{M^2\,dx}{2\,EI_r}$$

so that the bending energy calculated is less than that which is available during elastic-plastic deformation.

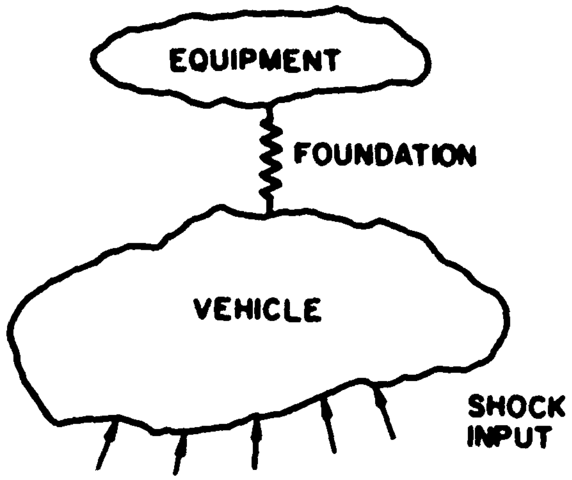


Fig. 1 — Equipment foundation system

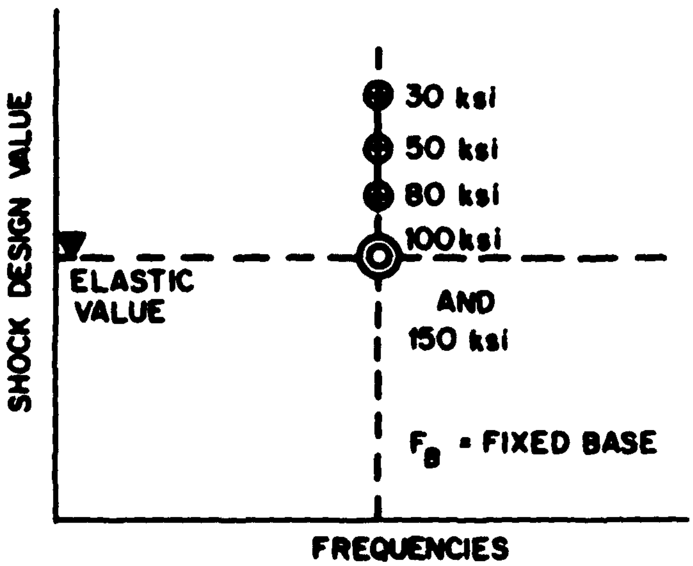


Fig. 2 — Shock design value versus fixed base frequency

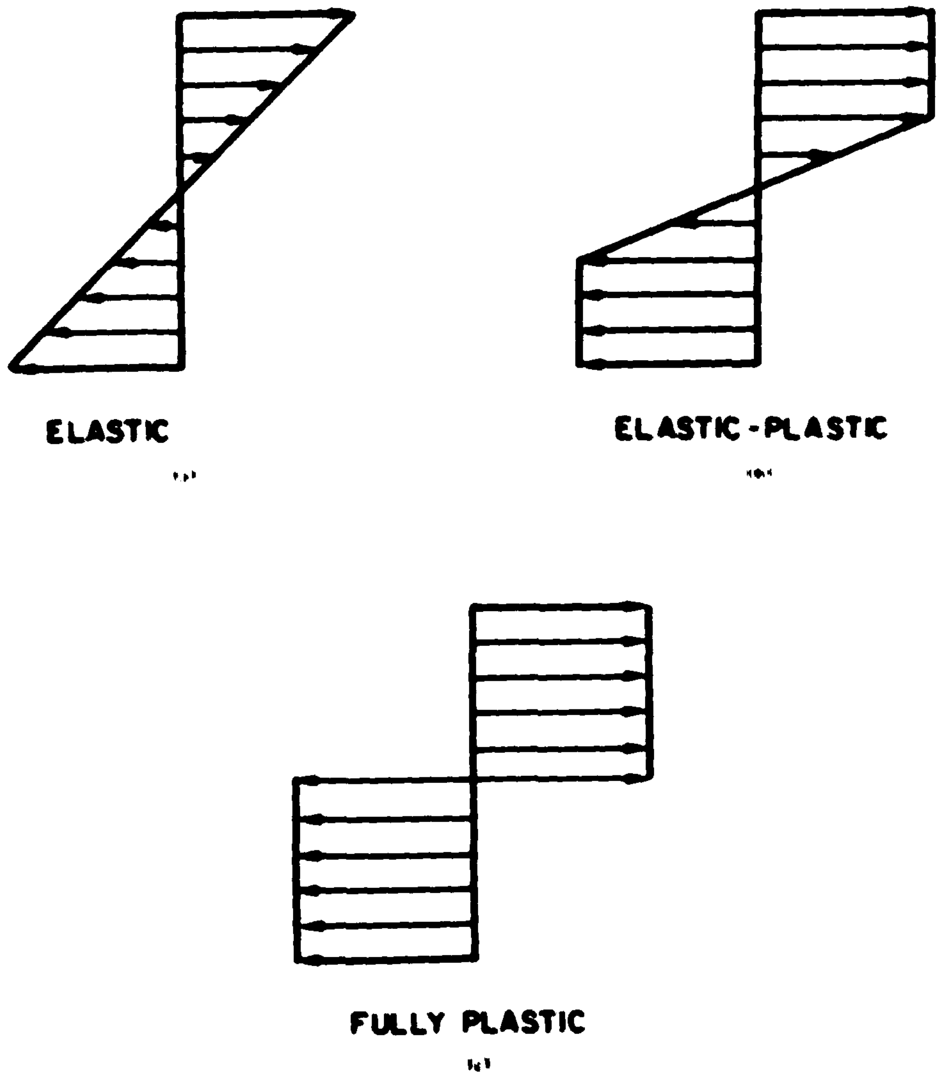
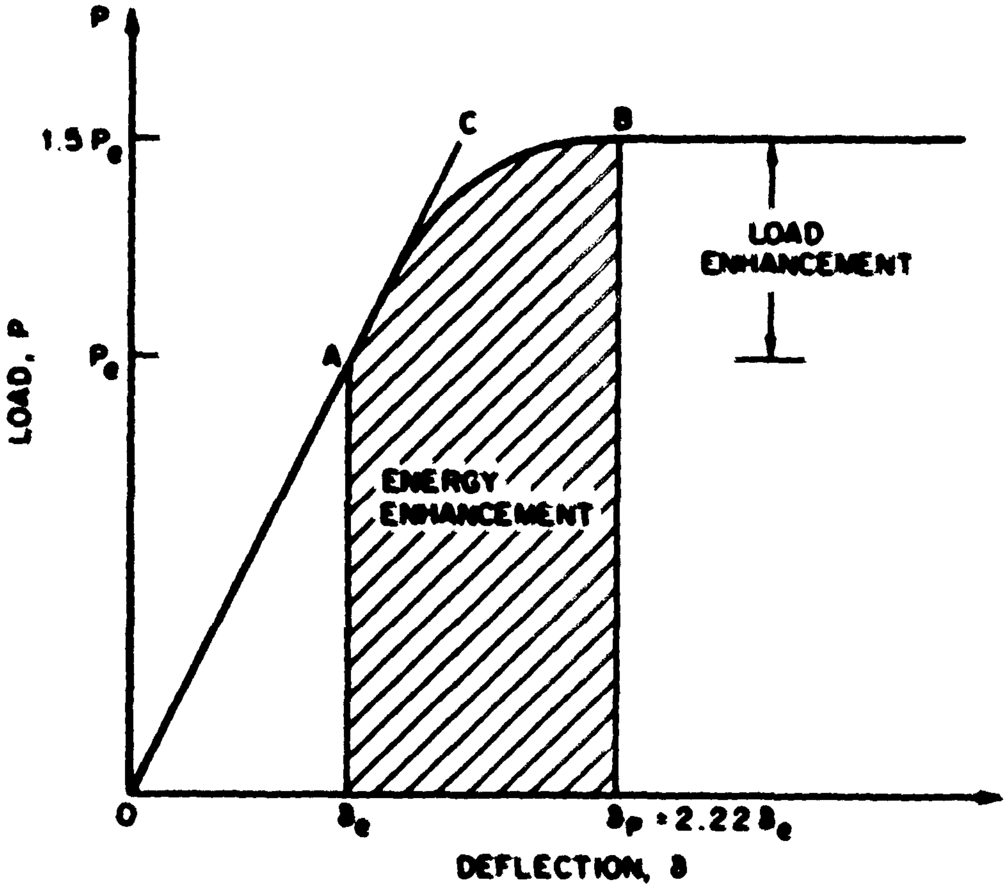


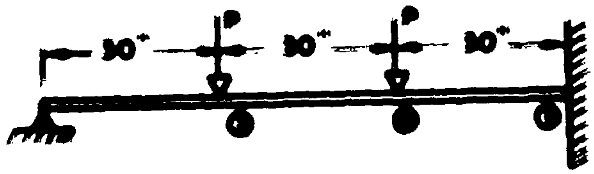
Fig. 1 — Stress distribution over beam cross-section

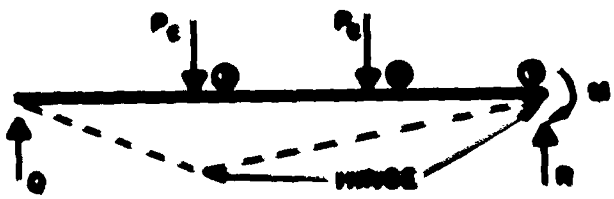
(a)



(b)

Fig. 4 — Cantilever Beam


30"
P
30"
P
30"


HINGE
R

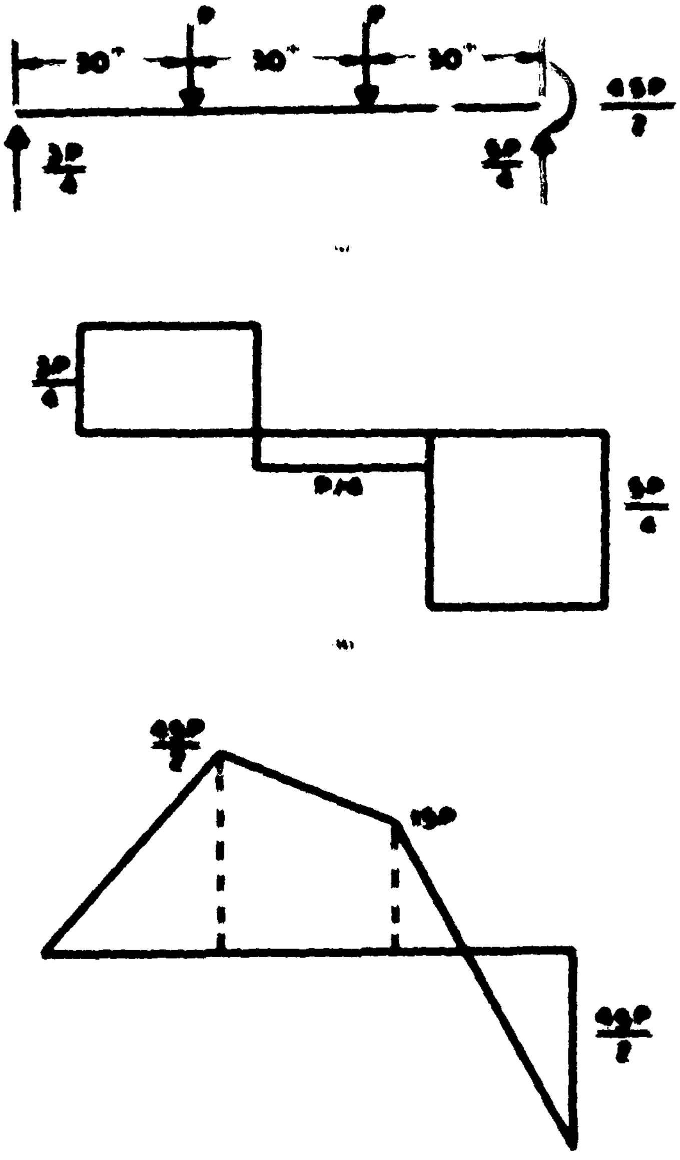
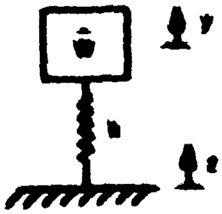


Fig. 7 – Load, shear and moment diagrams



$x = y - z$

Fig. 8 – Single degree of freedom system

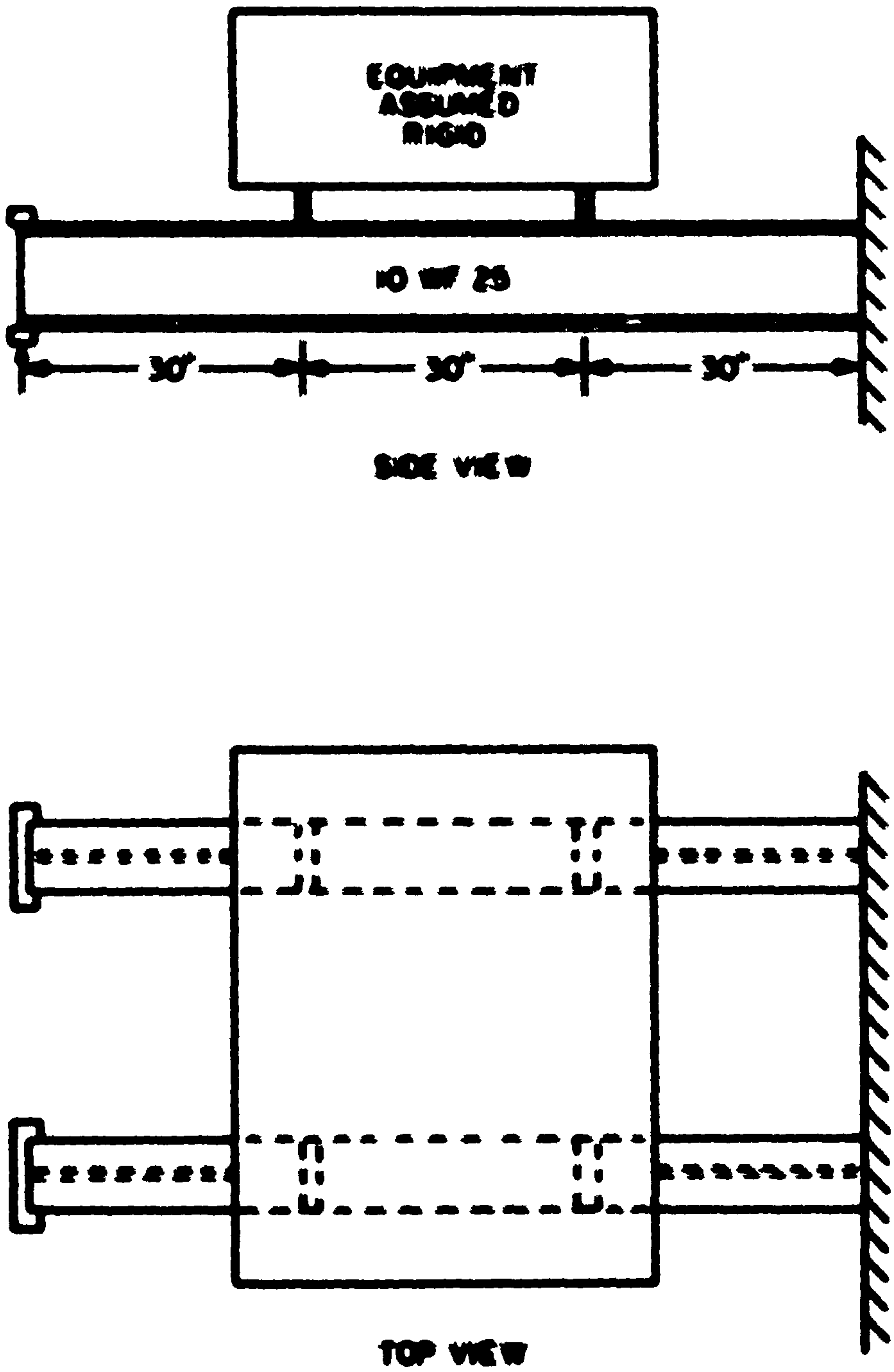
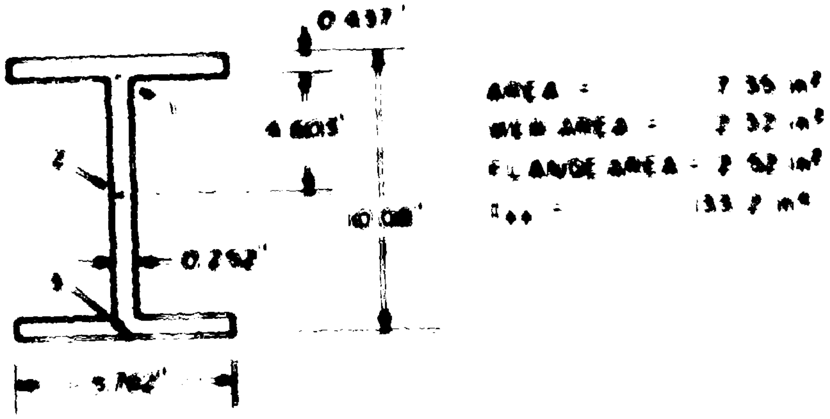
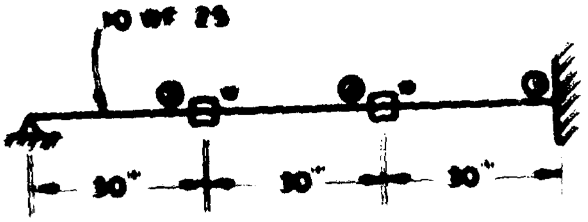


Fig. [illegible] — Equipment-foundation combination


0.437"
4.605"
10.08"
0.252"
5.762"
AREA = 7.35 in^2
WEB AREA = 2.32 in^2
FLANGE AREA = 2.52 in^2
I_{xx} = 133.2 in^4


10 WF 25
30"
30"
30"


30 Q
30"
Q
Q

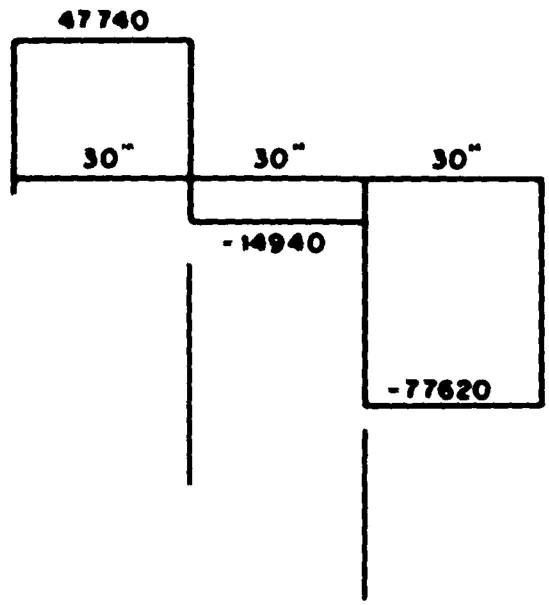


Fig. [illegible] — [illegible] diagram, pounds

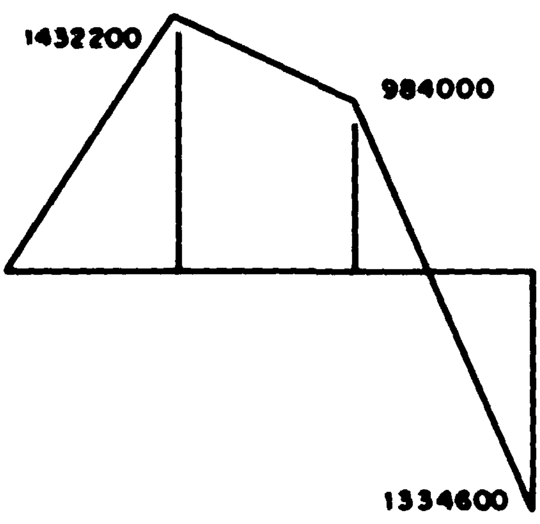


Fig. [illegible] — Moment diagram, inch pounds

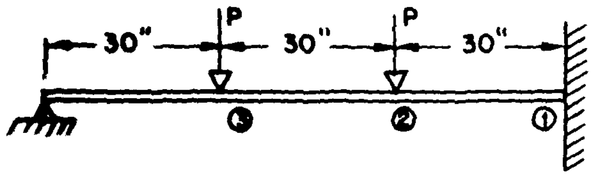


Fig. A-1 — Propped cantilever beam

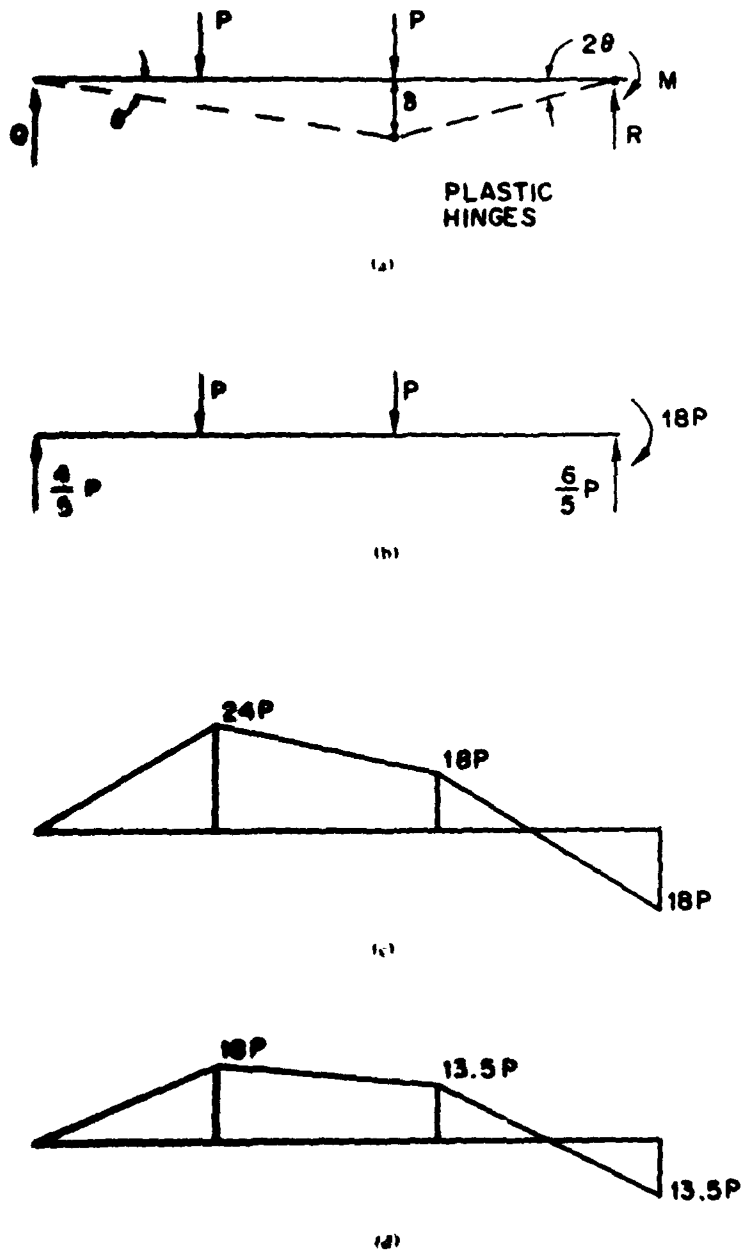


Fig. A-2 — (a) Failure mechanism candidate (b) loading diagram (c) inadmissible collapse moment diagram (d) statically admissible moment diagram

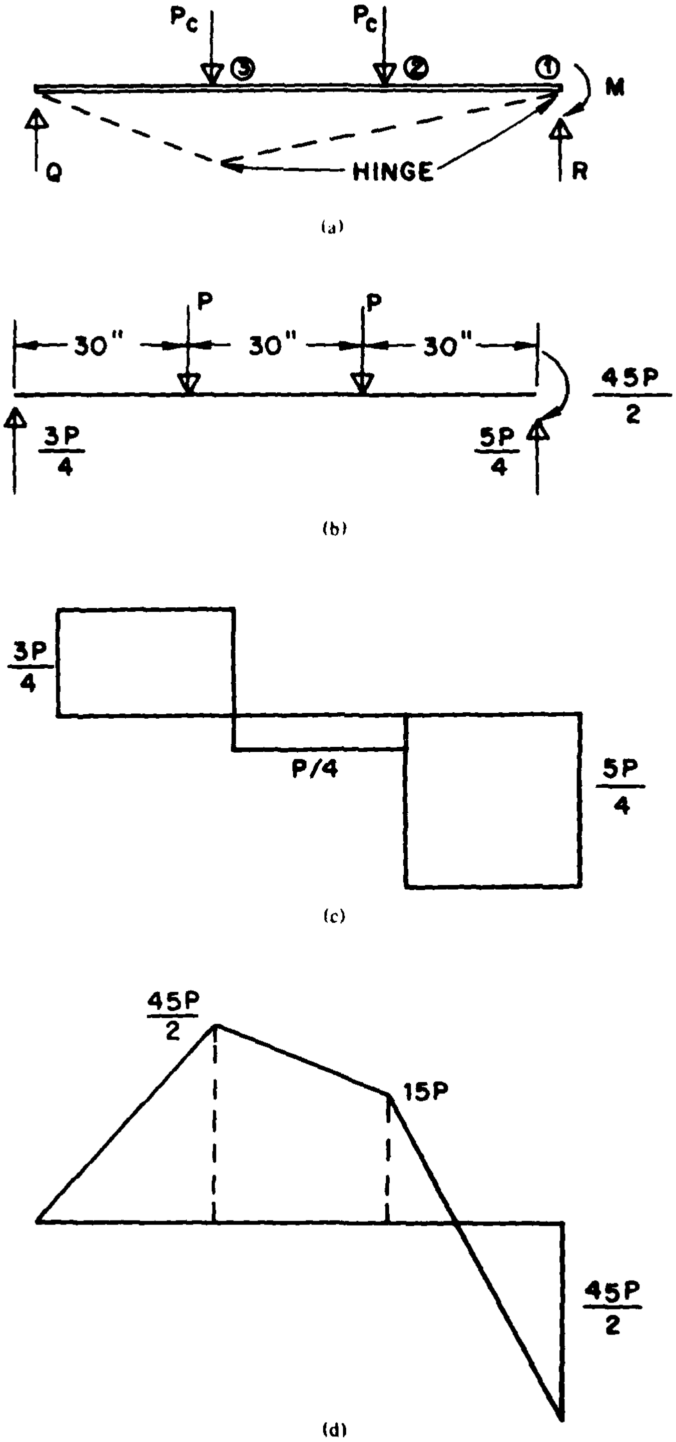


Fig. A-3 — (a) Failure mechanism candidate (b) loading diagram (c) shear diagram (d) moment diagram

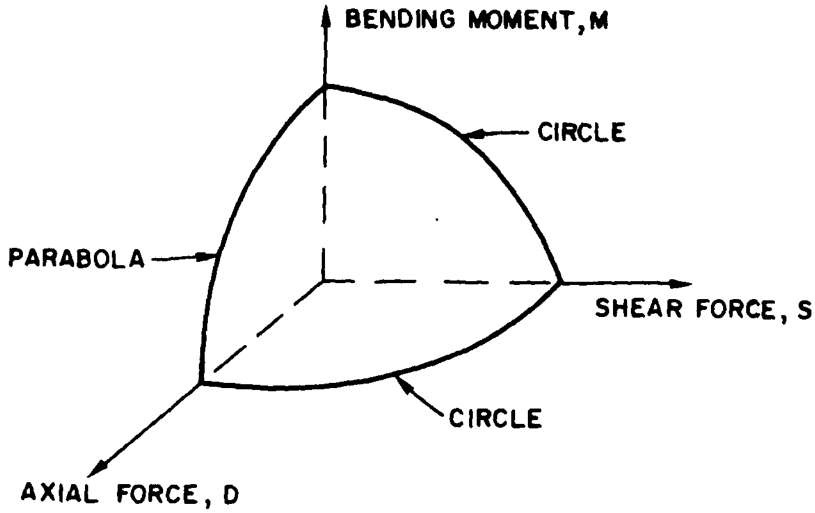


Fig. B-1 — Interaction surface for collapse

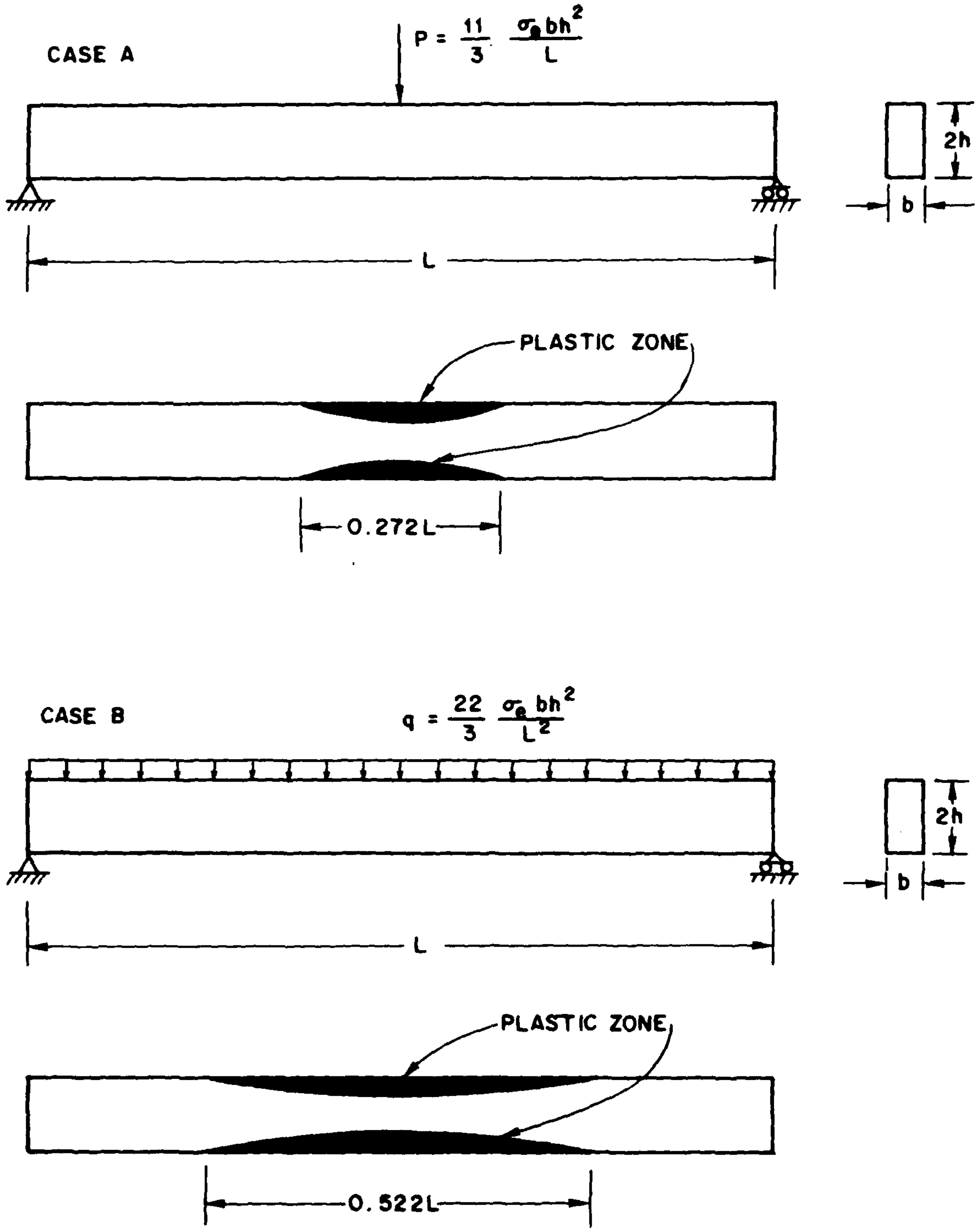


Fig. C-1 — Plastic zones for simply supported beams